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An Overdetermined Linear System

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1. Introduction

Consider a linear system of algebraic equations $Ax = b$, where $A = (a_{ij})$ is an $(n+1) \times n$ real matrix, $b = (b_i)$ an $(n+1)$-component real vector and $x = (x_i)$ an n-component real vector. Clearly, the system is overdetermined if b is not in the range of A and it is this problem to which we seek a "best" approximate solution. The technique employed is to impose an abstract norm on R^{n+1} and then find $x \in R^n$ such that the error vector $\eta(x) = b - Ax$ is minimized with respect to this norm. We refer to this question of finding such a solution with respect to a given abstract norm as "Problem (P)." When we use the standard l^p-norm, we let $\xi_p = (\xi_i{}^p)$ denote an l^p-solution to $Ax = b$, i.e.,

$$\| \eta(\xi_p)\|_p = \min_{x \in R^n} \| \eta(x)\|_p .$$

(The least-squares solution and the Tchebychev solution correspond to the values $p = 2$ and $p = \infty$, respectively.)

A set of vectors in R^m is said to satisfy the Haar condition if every set of m of them is linearly independent. Let $\mathscr{H}$ denote the set of all $(n+1) \times n$ matrices whose rows satisfy the Haar condition. In this paper, we concern ourselves with the problem of finding the l^p-solution to $Ax = b$ when $A \in \mathscr{H}$. In Section 2, we give an explicit formula for the l^p-solution and also discuss how it may be expressed as a convex combination of the solutions to the $n \times n$ subsystems. In Section 3, we allow b to be a random vector and demonstrate how one may favor a particular norm when observing a minimum variance criterion. Two examples of such stochastic overdetermined systems are given. Finally, some preliminary results pertaining to the matrix A are presented in the Appendix.

2. Explicit Forms of the l^p-Solution

Let A be an $(n+1) \times n$ matrix of rank n and A^{T} the transpose of A. Then $Ax = b$ has the unique l^2-solution

$$\xi_2 = (A^{\mathrm{T}}A)^{-1} A^{\mathrm{T}}b.$$

Moreover, the l^2-error vector is $s = \eta(\xi_2) = (I - A(A^{\mathrm{T}}A)^{-1} A^{\mathrm{T}})\, b$ (see [1]).

The dual norm $\|\cdot\|^{\sim}$ of a norm $\|\cdot\|$ on R^{n+1} is defined by

$$\| u \|^{\sim} = \max_{\|v\|=1} (u, v),$$

where

$$(u, v) = \sum_{i=1}^{n+1} u_i v_i$$

is the standard Euclidean inner product on R^{n+1}. Furthermore, $\tilde{u}$ is called a dual vector to $u \in R^{n+1}$ if $(\tilde{u}, u) = \| u \|^{\sim}$ and $\| \tilde{u} \| = 1$. From [4] we have the following

THEOREM (Sreedharan). *Let A be an $(n+1) \times n$ matrix of rank n and $s = \eta(\xi_2) = b - A\xi_2$ the l^2-error vector. If $s = 0$, then ξ_2 is a solution of Problem (P). If $s \neq 0$, then*

$$Ax = b - ((b, s)/\| s \|^{\sim})\, \tilde{s} \tag{2.1}$$

has a solution, and any solution of (2.1) *is a solution of Problem (P).*

From this point we shall only consider the case where R^{n+1} is equipped with the l^p-norm. Thus, the l^q-norm is the dual norm if $(1/p) + (1/q) = 1$. For $1 < p \leqslant \infty$, the dual vector of a nonzero vector $s = (s_i)$ is $\tilde{s}_p = (\tilde{s}_i{}^p)$, where

$$\tilde{s}_i{}^p = (| s_i |/\| s \|_q)^{q-1} \operatorname{sgn} s_i, \qquad i = 1, 2, \ldots, n+1. \tag{2.2}$$

For $p = 1$, we define $\rho = \{r_1, r_2, \ldots, r_t\} = \{r\colon 1 \leqslant r \leqslant n+1$ and $| s_r | = \max_i | s_i | = \| s \|_\infty\}$ and $\tilde{s}_1(r) = (\tilde{s}_i{}^1(r))$, where

$$\begin{aligned} \tilde{s}_i{}^1(r) &= \operatorname{sgn} s_r \qquad \text{if} \quad i = r, \\ &= 0 \qquad\qquad \text{if} \quad i \neq r. \end{aligned}$$

Then any convex combination of $\tilde{s}_1(r)$ with $r \in \rho$ is a dual vector of s for $p = 1$.

Clearly

$$\lim_{p\to 1^+} \tilde{s}_p = (1/t) \sum_{j=1}^{t} \tilde{s}_1(r_j) \tag{2.3}$$

is also a dual vector of s for $p = 1$.

With the given l^p-norm and $s \neq 0$, we may now write (2.1) as $Ax = b^{(p)}$ where

$$b^{(p)} = (b_i^{(p)}) = b - ((b, s)/\| s \|_q) \tilde{s}_p \tag{2.4}$$

and $\tilde{s}_p$ is given by (2.2) for $1 < p \leqslant \infty$, while any convex combination of $\tilde{s}_1(r)$ with $r \in \rho$ can define $\tilde{s}_1$. It is shown in [1, 2] that $Ax = b^{(p)}$ has a unique solution for each p, $1 \leqslant p \leqslant \infty$. Furthermore, each is a respective l^p-solution to $Ax = b$. The l^p-solution to $Ax = b$ is unique when $1 < p < \infty$ since the l^p-norm is strictly convex. From [2] we have that the l^1-solution is unique if and only if ρ is a singleton set ($t = 1$), and the l^∞-soution is unique if and only if the rows of A satisfy the Haar condition. If $Ax = b^{(p)}$ has a unique solution, then it can be found by solving $A^T Ax = A^T b^{(p)}$. Hence the l^p-solution can be obtained from

$$\xi_p = (A^T A)^{-1} A^T b^{(p)}. \tag{2.5}$$

Let A^k be the $n \times n$ matrix obtained from A by deleting the kth row, $D_k \equiv \det(A^k)$, the determinant of A^k,

$$\Delta_q \equiv \sum_{k=1}^{n+1} | D_k |^q$$

and

$$\sigma \equiv \sum_{k=1}^{n+1} (-1)^{k-1} b_k D_k .$$

Now $A \in \mathscr{H}$ implies that there exists a nonsingular $n \times n$ matrix P with $| \det(P)| = 1$ such that $AP = G$ where

$$G = \begin{pmatrix} g_1 & 0 & \cdots & 0 \\ 0 & g_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & g_n \\ h_1 & h_2 & \cdots & h_n \end{pmatrix}$$

with $g_i \neq 0$ and $h_i \neq 0$, $i = 1, 2, \ldots, n$. Let G^k be the matrix G with the kth row deleted. Then for $k = 1, 2, \ldots, n + 1$

$$A^k P = G^k$$

and

$$\hat{D}_k \equiv \det(G^k) = D_k \det(P).$$

Consider the $(n+1) \times (n+1)$ matrix formed by extending A in the following way:

$$(\alpha_{ij}) \equiv (A \mid \delta),$$

where $\delta \equiv ((-1)^{n+1-i} D_i)$ is an $(n+1)$-component column vector. Define $m_{ij} = (\text{cofactor of } \alpha_{ji})/\Delta_2$ for $i = 1,\ldots, n, j = 1,\ldots, n+1$, or more explicitly,

$$m_{ij} = \frac{(-1)^{i+j}}{\Delta_2} \begin{vmatrix} a_{11} & \cdots & a_{1,i-1} & a_{1,i+1} & \cdots & a_{1n} & (-1)^n D_1 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ a_{j-1,1} & \cdots & a_{j-1,i-1} & a_{j-1,i+1} & \cdots & a_{j-1,n} & (-1)^{n-j+2} D_{j-1} \\ a_{j+1,1} & \cdots & a_{j+1,i-1} & a_{j+1,i+1} & \cdots & a_{j+1,n} & (-1)^{n-j} D_{j+1} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ a_{n+1,1} & \cdots & a_{n+1,i-1} & a_{n+1,i+1} & \cdots & a_{n+1,n} & (-1)^0 D_{n+1} \end{vmatrix}. \tag{2.6}$$

THEOREM 2.1. *Let $A \in \mathscr{H}$.*

(i) *If $\sigma = 0$, then $Ax = b$ is solvable and has the unique solution $x = Mb$ where $M = (m_{ij})$ is defined by* (2.6).

(ii) *If $\sigma \neq 0$, then an l^p-solution to $Ax = b$ is $\xi_p = B_p b$, where $B_p = (\beta_{ij}(p))$ is defined by*

$$\beta_{ij}(p) = m_{ij} + \frac{(-1)^j D_j}{\Delta_q} \sum_{k=1}^{n+1} (-1)^{k-1} \frac{m_{ik} \mid D_k \mid^q}{D_k}, \qquad 1 < p \leqslant \infty,$$

$$= m_{ij} + \frac{(-1)^j D_j}{\tau} \sum_{k\in\lambda} (-1)^{k-1} \frac{m_{ik}}{D_k}, \qquad p = 1,$$

$(1/p) + (1/q) = 1$, $\lambda = \{l_1, l_2,\ldots, l_\tau\} = \{l\colon 1 \leqslant l \leqslant n+1$ and $\mid D_l \mid = \max_i \mid D_i \mid\}$, and m_{ij} is given by (2.6). *The l^1-solution given here has the property that $\lim_{p\to1^+} \xi_p = \xi_1$, hence ξ_p is continuous with respect to p for $1 \leqslant p \leqslant \infty$. Furthermore, ξ_p is the unique l^p-solution for $1 < p \leqslant \infty$ while the l^1-solution is unique if and only if λ is a singleton set ($\tau = 1$).*

Proof. Since $s = (I - A(A^TA)^{-1} A^T)\, b$, Theorem A.1 (cf. Appendix) implies that

$$s_i = (\sigma/\Delta_2)(-1)^{i-1} D_i, \qquad i = 1, 2,\ldots, n+1. \tag{2.7}$$

(i) $A \in \mathscr{H}$ implies that rank $(A) = n$ and thus A is one-to-one. Now b is in the range of A if and only if $s = 0$ which, in turn, is equivalent to $\sigma = 0$.

Hence $Ax = b$ is uniquely solvable in this case and it has the solution $x = (A^{T}A)^{-1} A^{T}b = Mb$.

(ii) For $\sigma \neq 0$, we see from (2.7) that s_i is proportional to D_i, so ρ and λ represent the same set. Furthermore,

$$(b, s) = \sigma^2/\Delta_2 . \tag{2.8}$$

From (2.2), (2.3), (2.7), (2.8), and (2.4) we obtain

$$b_j^{(p)} = b_j + \frac{\sigma}{\Delta_q} \frac{(-1)^j \mid D_j \mid^q}{D_j}, \qquad 1 < p \leqslant \infty; \tag{2.9a}$$

$$\begin{aligned} b_j^{(1)} &= b_j + (\sigma(-1)^j/\tau D_j), && j \in \lambda, \\ &= b_j && j \notin \lambda. \end{aligned} \tag{2.9b}$$

Theorem A.2 (cf. Appendix) and (2.5) yield

$$\xi_i{}^p = \sum_{j=1}^{n+1} m_{ij} b_j^{(p)}, \qquad i = 1, 2,..., n; \qquad 1 \leqslant p \leqslant \infty;$$

hence

$$\xi_i{}^p = \sum_{j=1}^{n+1} \beta_{ij}(p)\, b_j , \qquad i = 1, 2,..., n, \qquad 1 \leqslant p \leqslant \infty,$$

via (2.9) and the definition of σ. We see that $b_j^{(p)}$, $\beta_{ij}(p)$, and thus ξ_p are continuous with respect to p for $1 \leqslant p \leqslant \infty$ when $\tilde{s}_1$ is defined by (2.3).

COROLLARY 2.1. *Let $A \in \mathscr{H}$, $\sigma \neq 0$, and $\mid D_i \mid = c$, $i = 1, 2,..., n + 1$, for some c. Then ξ_p is independent of p for $1 \leqslant p \leqslant \infty$.*

Proof. This follows immediately from the definition of $\beta_{ij}(p)$.

Let us now consider the $n \times n$ subsystem of equations

$$A^k Z^k = b^k \tag{2.10}$$

for $k = 1, 2,..., n + 1$, where b^k is the vector b with the kth element deleted. Then (2.10) has a unique solution Z^k since A^k is nonsingular ($D_k \neq 0$) by the Haar condition. We would like to establish a relation between the l^p-solution ξ_p and the Z^k's. For each $k = 1, 2,..., n + 1$, the $n \times n$ system

$$G^k W^k = b^k \tag{2.11}$$

also has a unique solution $W^k = (w_i{}^k)$ since $\mid \hat{D}_k \mid = \mid D_k \mid \neq 0$.

We note that

$$G^k = \begin{bmatrix} g_1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & g_{k-1} & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & g_{k+1} & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & & 0 & 0 & 0 & \cdots & g_n \\ h_1 & \cdots & h_{k-1} & h_k & h_{k+1} & \cdots & h_n \end{bmatrix}.$$

Hence the first $n - 1$ equations of the system (2.11) yield

$$w_i^k = b_i/g_i \qquad \text{for} \quad i \neq k. \tag{2.12a}$$

By Cramer's rule and through expanding the determinant in the numerator about the kth column, we obtain

$$w_k^k = \frac{1}{\hat{D}_k} \sum_{j\neq k} (-1)^{k+j}\, b_j \left(-\frac{\hat{D}_j}{g_k}\right) = \frac{1}{D_k g_k} \sum_{j\neq k} (-1)^{k+j-1}\, b_j D_j\,. \tag{2.12b}$$

Lemma 2.1. (i) *If* $\sigma = 0$ *then* $(G^{\mathrm{T}}G)^{-1}\, G^{\mathrm{T}} b = W \equiv (b_i/g_i)$.

(ii) *If* $\sigma \neq 0$ *then*

$$\begin{aligned} (G^{\mathrm{T}}G)^{-1}\, G^{\mathrm{T}} b^{(p)} &= (1/\Delta_q) \sum_{k=1}^{n+1} |\, D_k\,|^q\, W^k, \qquad 1 < p \leqslant \infty, \\ &= (1/\tau) \sum_{k\in\lambda} W^k, \qquad\qquad p = 1, \end{aligned}$$

where $b^{(p)}$, λ *and* τ *were defined in Theorem* 2.1.

Proof. Let $(G^{\mathrm{T}}G)^{-1}\, G^{\mathrm{T}} \equiv (\gamma_{ij})$. From the proof of Theorem A.1 we have, for $i = 1, 2,..., n, j = 1, 2,..., n + 1$,

$$\begin{aligned} \gamma_{ij} &= \Big(\sum_{k\neq i} D_k^2\Big)\Big/(\Delta_2 g_i), \qquad & i = j, \\ &= (-1)^{i+j-1}\, D_i D_j/(\Delta_2 g_i), \qquad & i \neq j. \end{aligned} \tag{2.13}$$

(i) It follows from (2.13) and Lemma A.1 that $\sigma = 0$ implies

$$\sum_{j=1}^{n+1} \gamma_{ij} b_j = b_i/g_i\,, \qquad i = 1, 2,..., n.$$

(ii) From (2.9a), we have

$$\sum_{j=1}^{n+1} \gamma_{ij} b_j^{(p)} = (1/\Delta_q) \sum_{k=1}^{n+1} |\, D_k\,|^q\, c_{ik}\,, \qquad i = 1, 2,..., n,\ 1 < p \leqslant \infty,$$

where

$$c_{ik} = \sum_{j=1}^{n+1} \gamma_{ij} b_j + (\sigma(-1)^k \gamma_{ik}/D_k).$$

From (2.13), the above equation yields

$$\begin{aligned} c_{ik} &= \frac{1}{D_k g_k} \sum_{j \neq k} (-1)^{k+j-1} b_j D_j, \qquad & i = k, \\ &= b_i/g_i, & i \neq k. \end{aligned}$$

Hence $c_{ik} = w_i^k$ (cf. (2.12)). This completes the proof for the case $1 < p \leqslant \infty$, while the result for $p = 1$ follows by letting $p \to 1^+$.

THEOREM 2.2. *Let $A \in \mathscr{H}$ and, for each $k = 1, 2,..., n + 1$, let Z^k be the unique solution of $A^k Z^k = b^k$.*

(i) *If $\sigma = 0$, then $Ax = b$ is solvable and it has the unique solution $x = Z^k$, $k = 1, 2,..., n + 1$ (all the Z^k's being equal).*

(ii) *If $\sigma \neq 0$, then*

$$\xi_p = (1/\Delta_q) \sum_{k=1}^{n+1} | D_k |^q Z^k \tag{2.14}$$

is the unique l^p-solution to $Ax = b$ for $1 < p \leqslant \infty$. If λ and τ are defined as in Theorem 2.1, then

$$\xi_1 = (1/\tau) \sum_{k \in \lambda} Z^k \tag{2.15}$$

is an l^1-solution to $Ax = b$ which is unique if and only if λ is a singleton set ($\tau = 1$). Furthermore, the l^p-solution provided here is continuous with respect to p for $1 \leqslant p \leqslant \infty$.

Proof. Since A^k is nonsingular, it follows from $A^k P = G^k$, (2.10), and (2.11) that

$$Z^k = PW^k, \qquad k = 1, 2,..., n + 1. \tag{2.16}$$

(i) From (2.12), $\sigma = 0$ implies $w_i^k = b_i/g_i$, $i = 1, 2,..., n$, which is independent of k. Hence $W^k = W \equiv (b_i/g_i)$, $k = 1, 2,..., n + 1$, and $Z^k = PW^k = PW$, $k = 1, 2,..., n + 1$. From Theorem 2.1, $Ax = b$ is uniquely solvable with solution $x = (A^{\mathrm{T}}A)^{-1} A^{\mathrm{T}}b$. Therefore (A.1) and Lemma 2.1 show that $x = PW = Z^k$.

(ii) From (2.5) and (A.1) we have

$$\xi_p = P(G^TG)^{-1} G^T b^{(p)}. \tag{2.17}$$

Hence (2.14) and (2.15) follow from (2.17), Lemma 2.1, and (2.16).

Remark. Theorem 2.2 demonstrates how an l^p-solution of an overdetermined system with $A \in \mathscr{H}$ may be expressed as a convex combination of the solutions of the $n \times n$ subsystems.

3. A Stochastic Overdetermined System

Consider an overdetermined system $Ax = b$, where $A \in \mathscr{H}$ is a constant matrix but now $b = (b_i)$ is a random vector. Let $E(\cdot)$, $V(\cdot)$, and $\mathrm{Cov}(\cdot, \cdot)$ denote the expected value, variance, and covariance operators, respectively. We assume $V(b_i) < \infty$ for $i = 1, 2,..., n + 1$. From Theorem 2.1 we have

$$\xi_i^p = \sum_{j=1}^{n+1} \beta_{ij}(p)\, b_j, \qquad i = 1,..., n,\ 1 \leqslant p \leqslant \infty, \tag{3.1}$$

and hence

$$E(\xi_i^p) = \sum_{j=2}^{n+1} \beta_{ij}(p)\, E(b_j), \qquad i = 1,..., n,\ 1 \leqslant p \leqslant \infty. \tag{3.2}$$

We note that $E(\xi_p)$ is the l^p-solution to $Ax = E(b)$. We call $E(\xi_{\hat{p}})$ our "preferred approximate solution" to the stochastic overdetermined system if $\hat{p}$ is selected according to the minimum variance condition

$$v(\hat{p}) = \min_{1 \leqslant p \leqslant \infty} v(p),$$

where

$$v(p) = \sum_{i=1}^{n} V(\xi_i^p)$$

is the trace of the covariance matrix. Note that $v(p)$ is invariant under orthogonal transformations and is equal to the sum of the eigenvalues. Then, in a sense, it measures the total spread of the random variables. If $| D_i | = c$ for $i = 1, 2,..., n + 1$ then by Corollary 2.1 $v(p)$ is independent of p and thus any value of p, $1 \leqslant p \leqslant \infty$ will suffice for $\hat{p}$. Applying the variance operator to (3.1), we obtain

$$V(\xi_i^p) = \sum_{j=1}^{n+1} \beta_{ij}^2(p)\, V(b_j) + 2 \sum_{j<k} \beta_{ij}(p)\, \beta_{ik}(p)\, \mathrm{Cov}(b_j, b_k).$$

Therefore

$$v(p) = \sum_{j=1}^{n+1} V(b_j) \sum_{i=1}^{n} \beta_{ij}^2(p) + 2 \sum_{j<k} \operatorname{Cov}(b_j, b_k) \sum_{i=1}^{n} \beta_{ij}(p)\, \beta_{ik}(p). \tag{3.3}$$

We note that $v(p)$ is continuous for $1 \leqslant p \leqslant \infty$ and differentiable for $1 < p < \infty$.

Without loss of generality, we then assume that $| a_{11} | > | a_{21} |$ for the case $n = 1$. Here

$$M = (m_{11}, m_{12}) = \left(\frac{a_{11}}{a_{11}^2 + a_{21}^2}, \frac{a_{21}}{a_{11}^2 + a_{21}^2} \right),$$

$$\begin{aligned} \beta_{1j}(p) &= \frac{| a_{j1} |^q}{a_{j1}(| a_{11} |^q + | a_{21} |^q)}, && 1 < p \leqslant \infty, \\ &= (1 + (-1)^{j+1})/2a_{11}, && p = 1, \end{aligned} \qquad j = 1, 2,$$

and from (3.2) and (3.3)

$$\begin{aligned} E(\xi_p) &= \frac{| a_{11} |^q \dfrac{E(b_1)}{a_{11}} + | a_{21} |^q \dfrac{E(b_2)}{a_{21}}}{| a_{11} |^q + | a_{21} |^q}, && 1 < p \leqslant \infty, \\ &= E(b_1)/a_{11}, && p = 1, \\ v(p) &= \frac{K_1}{(| a_{11} |^q + | a_{21} |^q)^2}, && 1 < p \leqslant \infty, \\ &= V(b_1)/a_{11}^2, && p = 1, \end{aligned}$$

where

$$K_1 = | a_{11} |^{2q-2} V(b_1) + | a_{21} |^{2q-2} V(b_2) + 2 \frac{| a_{11}a_{21} |^q}{a_{11}a_{21}} \operatorname{Cov}(b_1, b_2).$$

If $V(b_1) = 0$ and $V(b_2) \neq 0$ we choose $\hat{p} = 1$, while if $V(b_1) \neq 0$ and $V(b_2) = 0$ we choose $\hat{p} = \infty$. For $V(b_1) \neq 0$ and $V(b_2) \neq 0$ we find $\lim_{p \to 1^+} (dv/dp) = \lim_{p \to \infty}(dv/dp) = 0$. Furthermore, $dv/dp = 0$ at $p = p_0$, where

$$\begin{aligned} p_0 &= 1, && \theta = 0, \\ &= \infty, && \theta = 1, \\ &= 1 + \frac{\ln(| a_{11} |/| a_{21} |)}{\ln | \theta |}, && \text{otherwise}, \end{aligned} \tag{3.4}$$

with

$$\theta = \frac{|a_{11}|[V(b_2) - (a_{21}/a_{11}) \operatorname{Cov}(b_1, b_2)]}{|a_{21}|[V(b_1) - (a_{11}/a_{21}) \operatorname{Cov}(b_1, b_2)]}.$$

Now $p_0 \geqslant 1$ if and only if $\theta \geqslant 1$, in which case

$$v(p_0) = \frac{V(b_1) V(b_2) - \operatorname{Cov}(b_1, b_2)}{a_{11}^2 V(b_2) + a_{21}^2 V(b_1) - 2a_{11}a_{21} \operatorname{Cov}(b_1, b_2)}.$$

If $\theta \geqslant 1$, we choose $\hat{p}$ such that $v(\hat{p}) = \min\{v(1), v(p_0), v(\infty)\}$; otherwise, we choose $\hat{p}$ such that $v(\hat{p}) = \min\{v(1), v(\infty)\}$. If, in addition, b_1 and b_2 are uncorrelated, then $\theta = (|a_{11}| V(b_2))/(|a_{21}| V(b_1))$ and we find explicitly that

$$\begin{aligned} \hat{p} &= \infty & 0 < \theta \leqslant 1 \\ &= 1 + \frac{\ln(|a_{11}|/|a_{21}|)}{\ln|\theta|} & \theta > 1. \end{aligned}$$

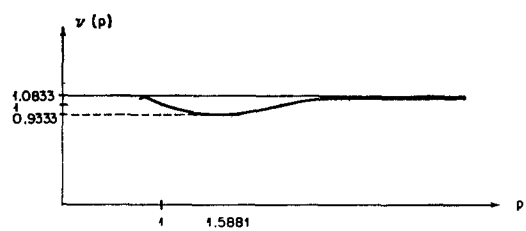
We now present two examples, the solutions of which were obtained utilizing a computer. $v(p)$ was calculated according to (3.3), thus deciding $\hat{p}$. The corresponding graph was also provided. Note that any type of distribution with $V(b_i) < \infty$ for $i = 1, 2, \ldots, n + 1$ and yielding the same first two moments would produce the same choice of $\hat{p}$.

EXAMPLE 1. $A = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and the joint distribution of b is

$b_1 \backslash b_2$	1	5
2	$\frac{3}{8}$	$\frac{1}{8}$
6	$\frac{1}{4}$	$\frac{1}{4}$

Hence $E(b_1) = 4$, $E(b_2) = 2.5$, $V(b_1) = 4$, $V(b_2) = 3.75$, $\operatorname{Cov}(b_1, b_2) = 1$. Here $\theta > 1$ and therefore (3.4) yields $p_0 = 1.5881$. Since $v(1) = 1$, $v(p_0) = 0.9333$ and $v(\infty) = 1.0833$, we choose $\hat{p} = 1.5881$. Hence our preferred approximate solution is

$$E(\xi_{1.5881}) = 2.0666,$$



EXAMPLE 2.

$$A = \begin{pmatrix} 0.5 & 1 & 0.5 \\ -3 & -9 & -6 \\ 4 & 12 & 6 \\ 2.5 & 6.5 & 3.5 \end{pmatrix}$$

and the joint distribution of b is $b_4 = 0$,

$b_3 = 4$

$b_1 \backslash b_2$	1	5
2	1/8	1/4
6	1/8	0

$b_3 = 6$

$b_1 \backslash b_2$	1	5
2	1/16	1/16
6	1/8	1/4

Thus

$$E(b_1) = 4, \quad E(b_2) = 3.25, \quad E(b_3) = 5, \quad E(b_4) = 0,$$

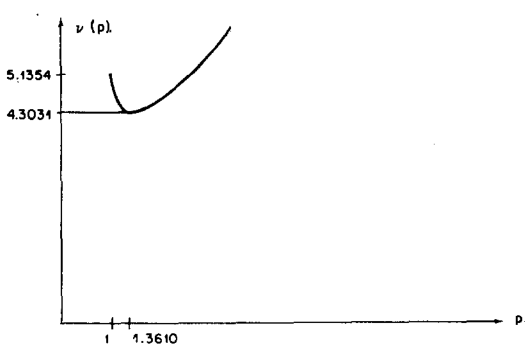
$$V(b_1) = 4, \quad V(b_2) = 3.9375, \quad V(b_3) = 1, \quad V(b_4) = 0,$$

$$\mathrm{Cov}(b_1, b_2) = -0.5, \quad \mathrm{Cov}(b_1, b_3) = 1, \quad \mathrm{Cov}(b_2, b_3) = 0.25,$$

$$\mathrm{Cov}(b_1, b_4) = \mathrm{Cov}(b_2, b_4) = \mathrm{Cov}(b_3, b_4) = 0.$$

Here $\hat{p} = 1.3610$,

$$E(\xi_{1.3610}) = \begin{pmatrix} -2.8273 \\ 3.6896 \\ -4.6631 \end{pmatrix},$$



APPENDIX: Some Results on Matrices

As a consequence of the Binet–Cauchy theorem [3], we have

Lemma A.1. *If A is an $(n+1) \times n$ matrix, then* $\det(A^{\mathrm{T}}A) = \Delta_2$. *We note that for $A \in \mathscr{H}$, $\Delta_2 \neq 0$.*

Theorem A.1. *If $A \in \mathscr{H}$, then*

$$I - A(A^{\mathrm{T}}A)^{-1} A^{\mathrm{T}}$$
$$= \frac{1}{\Delta_2} \begin{pmatrix} D_1{}^2 & -D_1D_2 & \cdots & (-1)^n D_1D_{n+1} \\ -D_2D_1 & D_2{}^2 & \cdots & (-1)^{n+1} D_2D_{n+1} \\ \vdots & \vdots & & \vdots \\ (-1)^n D_{n+1}D_1 & (-1)^{n+1} D_{n+1}D_2 & \cdots & (-1)^{2n} D_{n+1}^2 \end{pmatrix}$$
$$= \frac{\delta\delta^{\mathrm{T}}}{\Delta_2}.$$

Proof. Since $AP = G$, it follows that

$$(A^{\mathrm{T}}A)^{-1} A^{\mathrm{T}} = P(G^{\mathrm{T}}G)^{-1} G^{\mathrm{T}} \tag{A.1}$$

and

$$A(A^{\mathrm{T}}A)^{-1} A^{\mathrm{T}} = G(G^{\mathrm{T}}G)^{-1} G^{\mathrm{T}}. \tag{A.2}$$

From the definition of G^k, we see that

$$\begin{aligned} \hat{D}_k &= (-1)^{n-k} h_k \prod_{i \neq k} g_i, \qquad & k &= 1, 2, \ldots, n, \\ &= \prod_{i=1}^{n} g_i, & k &= n+1. \end{aligned}$$

Since $\hat{D}_k = D_k \det(P)$ and $|\det(P)| = 1$, we have $D_iD_j = \hat{D}_i\hat{D}_j$ for $i, j = 1, 2, \ldots, n+1$. From Lemma A.1,

$$\det(A^{\mathrm{T}}A) = \sum_{k=1}^{n+1} D_k{}^2 = \sum_{k=1}^{n+1} \hat{D}_k{}^2 = \sum_{k=1}^{n} h_k{}^2 \prod_{i \neq k} g_i{}^2 + \prod_{i=1}^{n} g_i{}^2.$$

Since

$$G^{\mathrm{T}}G = \begin{pmatrix} g_1{}^2 + h_1{}^2 & h_1h_2 & \cdots & h_1h_n \\ h_2h_1 & g_2{}^2 + h_2{}^2 & \cdots & h_2h_n \\ \vdots & \vdots & & \vdots \\ h_nh_1 & h_nh_2 & \cdots & g_n{}^2 + h_n{}^2 \end{pmatrix},$$

we have

$$(G^{\mathrm{T}}G)^{-1} =$$

$$\frac{1}{\Delta_2}\begin{bmatrix} \Big(\sum_{k\neq 1}\frac{h_k^2}{g_n^2}+1\Big)\prod_{i\neq 1} g_i^2 & -h_1h_2\prod_{i\neq 1,2} g_i^2 & \cdots & -h_1h_n\prod_{i\neq 1,n} g_i^2 \\ -h_1h_2\prod_{i\neq 2,1} g_i^2 & \Big(\sum_{k\neq 2}\frac{h_k^2}{g_k^2}+1\Big)\prod_{i\neq 2} g_i^2 & \cdots & -h_2h_n\prod_{i\neq 2,n} g_i^2 \\ \vdots & \vdots & & \vdots \\ -h_nh_1\prod_{i\neq n,1} g_i^2 & -h_nh_2\prod_{i\neq n,2} g_i^2 & \cdots & \Big(\sum_{k\neq n}\frac{h_k^2}{g_k^2}+1\Big)\prod_{i\neq n} g_i^2 \end{bmatrix}.$$

Hence

$$G(G^{\mathrm{T}}G)^{-1}\, G^{\mathrm{T}} =$$

$$\frac{1}{\Delta_2}\begin{bmatrix} \sum_{k\neq 1} D_k^2 & D_1D_2 & -D_1D_3 & \cdots\ (-1)^{n+1}D_1D_{n+1} \\ D_2D_1 & \sum_{k\neq 2} D_k^2 & D_2D_3 & \cdots\ (-1)^{n+2}D_2D_{n+1} \\ \vdots & \vdots & \vdots & \vdots \\ (-1)^{n+1}D_{n+1}D_1 & (-1)^{n+2}D_{n+1}D_2 & (-1)^{n+3}D_{n+1}D_3\ \cdots & \sum_{k\neq n+1} D_k^2 \end{bmatrix}.$$

The result now follows in view of Lemma A.1 and (A.2).

LEMMA A.2. *If* $A \in \mathscr{H}$ *and* M_1 *and* M_2 *are both* $n \times r$ *matrices then* $AM_1 = AM_2$ *implies that* $M_1 = M_2$.

THEOREM A.2. *If* $A \in \mathscr{H}$, *then* $(A^{\mathrm{T}}A)^{-1}\, A^{\mathrm{T}} = M = (m_{ij})$, *where* m_{ij} *is defined in* (2.6).

Proof. By Lemma A.2, this theorem is proved if $AM = A(A^{\mathrm{T}}A)^{-1}\, A^{\mathrm{T}}$. The entry of AM at the lth row and the jth column is

$$\sum_{i=1}^{n} a_{li}m_{ij} = \frac{1}{\Delta_2}\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} & (-1)^n\, D_1 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{j-1,1} & a_{j-1,2} & \cdots & a_{j-1,n} & (-1)^{n-j+2}\, D_{j-1} \\ a_{l,1} & a_{l,2} & \cdots & a_{l,n} & 0 \\ a_{j+1,1} & a_{j+1,2} & \cdots & a_{j+1,n} & (-1)^{n-1}\, D_{j+1} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{n+1,1} & a_{n+1,2} & \cdots & a_{n+1,n} & (-1)^0\, D_{n+1} \end{vmatrix},$$

where $l = 1,..., n + 1, j = 1,..., n + 1$.

If $l = j$, the determinant expanded about the last column yields

$$\sum_{i=1}^{n} a_{ji} m_{ij} = (1/\Delta_2) \sum_{i \neq j} D_i^2.$$

If $l \neq j$, the same expansion yields zero for each entry except the lth one, for which we have

$$\sum_{i=1}^{n} a_{li} m_{ij} = ((-1)^{l+j-1}/\Delta_2)\, D_l D_j\,.$$

Therefore, we see that matrix AM is precisely $G(G^{\mathrm{T}}G)^{-1}\, G^{\mathrm{T}} = A(A^{\mathrm{T}}A)^{-1}\, A^{\mathrm{T}}$ as constructed in Theorem A.1.

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